BE 159 Winter 2017
Homework #5
Due at the start of class, March 1, 2017

Problem 1 (Spontaneous flow with patterning (50 points))**.**
In this problem, we will investigate how patterns with flow can spontaneously emerge from a active fluid. Specifically, we will consider a one-dimensional system with a single chemical species of concentration $c(x,t)$ that obeys a simple advection-diffusion equation.

$$\partial_t c = D\partial_x^2 c - \partial_x(vc), \tag{1}$$

where v is the fluid flow velocity in the x-direction. The fluid is active, like a discussed in the Mayer, et al. paper, with the same governing equation (which you derived in the last homework),

$$\partial_x \sigma_a = -\eta\partial_x^2 v + \gamma v. \tag{2}$$

We can already see that the two dynamical equations are coupled because the fluid flow velocity appears in both equations. The situation becomes more interesting when the active stress depends on the concentration of the chemical species, i.e., it is a regulator of active stress. (We will henceforth call the chemical species a "regulator.") We therefore define

$$\sigma_a = \sigma_a^0 f(c), \tag{3}$$

where $f(c)$ is a dimensionless function of c and σ_a^0 is the scale of the active stress.

a) We will assume no-flux boundary conditions, i.e., $-D\partial_x c + vc = 0$ at $x = 0$ and $x = L$. Discuss why or show that there exists a single homogeneous steady state, which we will arbitrarily call $c = c_0$.

b) Show that the steady state velocity is related to the steady state concentration by $v = D\partial_x \ln c$. This implies that a non-homogeneous profile has fluid flow, and also that there is no flow for the homogeneous steady state.

c) The **Péclet number**, Pe, is the ratio of the diffusive to advective time scales.

   i) Explain when $\text{Pe} = \sigma_a^0/D\gamma$ is a good definition of the Péclet number for this case.

   ii) If $\text{Pe} \gg 1$, does advective or diffusive transport dominate the dynamics, and why?

d) We will now perform a linear stability analysis about the homogeneous quiescent steady state. The goal is to see if the system can spontaneously form a non-homogeneous pattern in $c(x)$ accompanied by flow.

   i) Consider a small perturbation to the homogenous steady state, $c = c_0 + \delta c$, with $\delta c = \delta c_0\, \mathrm{e}^{st+ikx}$, where k is the wave number of the spatial perturbation and $s = s(k)$. Show that to linear order in δc that

$$v(x) = \frac{ik\sigma_a^0 f_0}{\gamma + \eta k^2}\,\delta c, \tag{4}$$

   where $f_0 = \partial_c f(c_0)$, the value of the first derivative of $f(c)$ with respect to c, evaluated at c_0.

ii) Show that the homogenous steady state is unstable (and therefore patterns may spontaneously form) if

$$\frac{\mathrm{Pe}\, c_0 f_0}{1 + (\pi\ell/L)^2} > 1, \tag{5}$$

where ℓ is the familiar hydrodynamic length scale from the Mayer paper, $\ell = \sqrt{\eta/\gamma}$. *Hint*: Because of the geometry of the system, allowed wave numbers are $k = n\pi/L$, where n is an integer.

iii) What does equation (5) say about the requirements of the functional form of $f(c)$ in order to get patterns?

e) Presumably, the concentration of the active stress regulator can be controlled via regulation of gene expression. Similarly, other parameters like σ_a^0, η, and γ are not as easily controlled. In light of this, discuss the repercussions of equation (5) for how flowing patterns may be turned on and off. *Hint*: You need to think about the functional form of $f(c)$. It might help to think of it having a Hill-like form, such as $f(c) = c/(1 + c)$, where c is now a dimensionless concentration.

f) Sketch a plot of a peak in concentration of the regulator along with the fluid flow velocity that we might see at steady state. It need not be quantitatively accurate; just a sketch will do. Based on this sketch, give a qualitative physical discussion on how patterns can be maintained through this active advection-diffusion mechanism.

This problem has shown you how the coupling of diffusion and biochemical regulation of active stress can give patterns in development. This is very much a thought experiment similar to Turing's. These thought experiments, though they often do not describe *exactly* how a biological phenomenon happens, are still useful and informative.

Problem 2 (Flow patterns in *C. elegans* (25 points))**.**
In the Mayer, et al paper, we studied cortical flow in the one-cell *C. elegans* embryo. The cortex is in contact with the cytoplasm. If the cytoplasm is a viscous fluid, we might expect that the movement of the cortex will drive flow in the cytoplasm.

a) Make a sketch of the flow we would expect in the cytoplasm.

b) Imagine we take a cross section at the center of the one-cell *C. elegans* embryo that is orthogonal to the anterior-posterior axis. What is the *net* flow of cytoplasm that flows through this cross section?

Problem 3 (Hydrodynamic coupling (25 points))**.**
Say I have two beads or radius a (say of order one micron) next to each other in a very viscous fluid, such that the distance between them is not too big, say of order a. The bead on the right is ferromagnetic, but the one on the left is not.

a) If I pull the ferromagnetic bead to the right using a magnet, what happens to both beads?

b) Now, say the ferromagnetic bead moves leftward. What happens to both beads?

c) Repeat (a), except with the beads now embedded in an elastic medium.

d) Why am I asking you this? In other words, what consequences might the physics exposed by these toy questions have on developmental processes?