BE 159 Winter 2018
Homework #5
Due at the start of class, February 28, 2018

Problem 5.1 (Equation of motion for the cortex).
In the Mayer, et al. paper, the authors describe the dynamics of cortical flow using the following equation.

$$-\eta\,\partial_x^2 v_x + \gamma v_x = \partial_x C. \tag{5.1}$$

Note that when we have repeated indices that are x, y, or z, summation is *not* assumed. In the notation we have been using in lecture, this is

$$-\eta\,\partial_x^2 v_x + \gamma v_x = \partial_x \sigma_a. \tag{5.2}$$

This equation describes how a gradient in active stress drives cortical flow against viscous dissipation and frictional losses. In this problem, you will derive this equation. Even though the nematic order does not appear in the equation, it is necessary in its derivation. We saw in lecture that we must have anisotropy to be able to support active stresses; this will become clear when we consider the nematic order explicitly in deriving the above cortical equation of motion.

In lecture, we defined the nematic order parameter as

$$Q_{ij} = S\left(n_i n_j - \frac{1}{3}\,\delta_{ij}\right). \tag{5.3}$$

Here, S is the magnitude of the local order. We wrote the active stress as a Taylor series expansion of the nematic order parameter as

$$\sigma_{\text{active}} = \sigma_a^0 \delta_{ij} + \sigma_a Q_{ij}. \tag{5.4}$$

Then, the stress tensor for a three-dimensional active nematic viscous fluid, which is how are are modeling the cortex, is

$$\sigma_{ij} = -\Pi\,\delta_{ij} + 2\eta\, v_{ij} + \sigma_{ij}^{\text{nematic}} + \sigma_a Q_{ij}, \tag{5.5}$$

where $\Pi = p - \sigma_a^0$ and v_{ij} is the symmetric part of the velocity gradient tensor,

$$v_{ij} = \frac{1}{2}\left(\partial_i v_j + \partial_j v_i\right). \tag{5.6}$$

We denote by $\sigma_{ij}^{\text{nematic}}$ the passive stresses due to nematic order. The $\sigma_{ij}^{\text{nematic}}$ term is directional active stress exerted along the nematic order. The equation of motion is then, considering again the interialess limit for an incompressible fluid,

$$\partial_j \sigma_{ij} = 0 = -\partial_i \Pi + \eta\,\partial_j \partial_j v_i + \partial_j \sigma_{ij}^{\text{nematic}} + \partial_j(\sigma_a Q_{ij}). \tag{5.7}$$

Starting from these equations, you will derive the equation of motion for the cortex, (5.2).

a) To simplify things, we will assume that the alignment of filaments in the cortex rapidly relax to equilibrium so that the dynamics of the order parameter may be neglected, i.e., Q_{ij} is constant. To find the equilibrium value of Q_{ij}, we note that the deformation energy of a nematic liquid crystal can be approximately written as

$$F_d = F_0 + \frac{\chi}{2} Q_{ij} Q_{ij} + \frac{L}{2} \left(\partial_k Q_{ij}\right) \left(\partial_k Q_{ij}\right). \tag{5.8}$$

Give an explanation as to why this is a reasonable functional form for the deformation free energy.[1]

We can find $\sigma_{ij}^{\text{nematic}}$ to be related to the functional derivative of the deformation energy. You do not need to derive this, but you get the result we stated in lecture.

$$\sigma_{ij}^{\text{nematic}} = \beta_1 \left(\chi - L \partial_k \partial_k\right) Q_{ij}. \tag{5.9}$$

b) The cortex is essentially a two-dimensional object. It is only about one micron thick, but has extent of over 50 microns. We therefore assume that the filaments of the cortex are aligned only in the x-y plane. In other words, $n_z = 0$, which means that $Q_{xz} = Q_{yz} = 0$ and $Q_{zz} = -S/3$. Given that it is constrained to two dimensions, find the value of Q_{ij} that minimizes the deformation free energy, subject to the constraint that alignment is confined to a thin sheet, i.e., that $n_z \approx 0$. Your result will be linear in S.

c) Next, we specify that the cortex does not bend or buckle, so the stresses normal to the two-dimensional cortex should vanish. In other words, $\sigma_{zz} = 0$. Based on this assumption, derive an expression for Π. *Hint:* Don't forget that in three dimensions, the material is incompressible, so $\partial_i v_i = 0$.

d) Using the expression you derived in part (c), along with the assumption that Q_{ij} is constant, show that the two dimensional equations of motion are

$$\eta \partial_z^2 v_x + 3\eta \partial_x^2 v_x + \eta \left(\partial_y^2 v_x + 2\partial_x \partial_y v_y\right) + \partial_x \sigma_a = 0, \tag{5.10}$$

$$\eta \partial_z^2 v_y + 3\eta \partial_y^2 v_y + \eta \left(\partial_x^2 v_y + 2\partial_y \partial_x v_x\right) + \partial_y \sigma_a = 0, \tag{5.11}$$

where we have absorbed a factor of $S/2$ into σ_a.

e) Show that

$$\eta \partial_z^2 v_x = \partial_z \sigma_{xz} + \eta \partial_x (\partial_x v_x + \partial_y v_y). \tag{5.12}$$

[1]There are also deep arguments about symmetry that come into play here, but you do not need to worry too much about those. You can read more about this particular form of the free energy; it is called a Landau-de Gennes expansion.

A similar relation holds in the y-direction. As a result, we have

$$\partial_z \sigma_{xz} + \eta\,(\partial_x^2 + \partial_y^2) v_x + 3\eta\,\partial_x(\partial_x v_x + \partial_y v_y) + \partial_x \sigma_a = 0, \tag{5.13}$$

$$\partial_z \sigma_{yz} + \eta\,(\partial_x^2 + \partial_y^2) v_y + 3\eta\,\partial_y(\partial_x v_x + \partial_y v_y) + \partial_y \sigma_a = 0. \tag{5.14}$$

f) Next, we will average the two equations over the thin dimension, z. That is, we will apply the operation $h^{-1}\int_0^h \mathrm{d}z$, where h is the cortical thickness, to each equation. We define

$$\bar{a} = \frac{1}{h}\int_0^h \mathrm{d}z\, a, \tag{5.15}$$

where a is some physical quantity, such as v_x. Show that if $\partial_x h \approx 0$, i.e., if h is approximately constant, then $\overline{\partial_x a} \approx \partial_x \bar{a}$. Going forward, you may assume that similar results hold for $\overline{\partial_y a}$, $\overline{\partial_x \partial_y a}$, and so on.

g) Perform the averages over equations (5.13) and (5.14). You will be left with a term like $h^{-1}\,\sigma_{xz}\big|_0^h$. Explain why we can write

$$\frac{1}{h}\;\sigma_{xz}\big|_0^h = -\gamma\,\bar{v}_x. \tag{5.16}$$

What is the meaning of the parameter γ?

h) Your equations should now look like

$$-\gamma\,\bar{v}_x + \eta\,(\partial_x^2 + \partial_y^2)\bar{v}_x + 3\eta\,\partial_x(\partial_x \bar{v}_x + \partial_y \bar{v}_y) + \partial_x \bar{\sigma}_a = 0, \tag{5.17}$$

$$-\gamma\,\bar{v}_y + \eta\,(\partial_x^2 + \partial_y^2)\bar{v}_y + 3\eta\,\partial_y(\partial_x \bar{v}_x + \partial_y \bar{v}_y) + \partial_y \bar{\sigma}_a = 0. \tag{5.18}$$

Explain why the quantity 3η is a two-dimensional bulk viscosity.

i) Now, we will assume that we can neglect curvature and that we have azimuthal symmetry in the *C. elegans* cortex. Under these assumptions, write a simplified version of equation (5.17). Is equation (5.18) still necessary? Finally, what do we need to do to get the final result we are after, equation (5.2)?