BE 159 Winter 2020
Homework #4
Due at the start of class, February 26, 2020

Problem 4.1 (Flow patterns in a cavity (15 points)).
A classic problem in beginning fluid mechanics courses features a channel with a lid sliding over it, as depicted below.

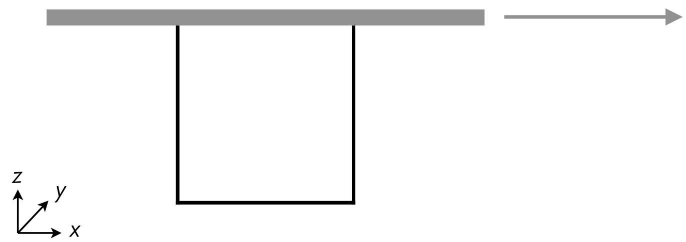

Figure 1: Depiction for a channel with a lid sliding over it. The channel and lid are very long in the y-direction.

a) Assuming the lid is sliding slowly enough such that the Reynolds number is very small, sketch the streamlines of the flow in the channel.
b) Imagine a cross-section of the channel in the y-z plane. What is the *net* flux of fluid through this cross-section as the lid is slid?

Problem 4.2 (Flow past and object (15 points)).
Take a look at the picture below of a cylinder moving in a tank of water. Is the Reynolds number above or below unity? Explain your reasoning.

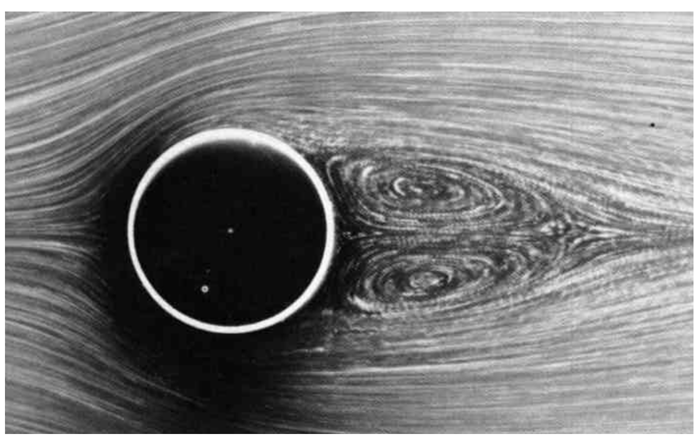

Figure 2: Photograph by Sadatoshi Taneda a cylinder moving through a tank of water. The flow is visualized using aluminum powder. The image is taken from *An Album of Fluid Motion* by Milton Van Dyke, Parabolic Press, 1982.

Problem 4.3 (Hydrodynamic coupling (20 points)).

Say I have two beads or radius a (say of order one micron) next to each other in a very viscous fluid, such that the distance between them is not too big, say of order a. The bead on the right is ferromagnetic, but the one on the left is not.

a) If I pull the ferromagnetic bead to the right using a magnet, what happens to both beads?
b) Now, say the ferromagnetic bead moves leftward. What happens to both beads?
c) Repeat (a), except with the beads now embedded in an elastic medium.
d) Why am I asking you this? In other words, what consequences might the physics exposed by these toy questions have on developmental processes?

Problem 4.4 (The reciprocal theorem (25 points)).
The reciprocal theorem is another very useful property of Stokes flow. (It is an analog to Lorentz reciprocal theorem from electromagnetism and is used in much the same way.) One way to state the theorem is as follows. Let v_i be a solution to the Stokes equations with stress tensor σ_{ij}. Let v'_i be another solution to the Stokes equations (naturally with different boundary conditions) with stress tensor σ'_{ij}. Then,

$$\partial_j(v'_i \sigma_{ij}) = \partial_j(v_i \sigma'_{ij}). \tag{4.1}$$

This means that if we can solve Stokes's equations for some set of boundary conditions, we can use that solution to get the solution for another, perhaps more difficult to work with, set of boundary conditions.

Derive the reciprocal theorem.

Problem 4.5 (Use of Green's functions (10 points)).
Explain in words why it is useful to use Green's functions of the Stokes equations to construct solutions of the Stokes equations in more complicated geometries.

Problem 4.6 (Limitation of using Stokes flow (15 points)).
In lecture 11, considered an object moving through a viscous fluid and deduced that hydrodynamic forces are long-ranged, since the fluid velocity decays away like $v(r) \sim r^{-1}$, where r is the distance from the object. We discovered this by enforcing that the total momentum flux through any spherical shell centered on the moving object must be the same. Note that in computing this momentum flux, we neglected kinetic energy, assuming the Reynolds number is small.

Given that the velocity decays away like r^{-1}, write an expression for the kinetic energy per volume, e_k of the fluid. Then use this expression to demonstrate that the total kinetic energy of the entire fluid volume (which you may assume to have infinite extent in space) is infinite.

This result shows that Stokes equation cannot be valid far from the moving body, and that a correction is needed. However, close to the moving body, Stokes equations are very good approximations at low Reynolds number.