BE 159 Winter 2021
Homework #6
Due at 5 pm, March 14, 2021

Problem 6.1 (Spontaneous flow with patterning).
The importance of mechanochemical coupling between the cortex and the PAR proteins, which also serve to regulate cortical active stress, was emphasized in the Gross paper. The full model presented in the Gross paper including explicit modeling of the reaction-diffusion-advection dynamics of the PAR complexes and myosin. In this problem, we will strip away much of that detail (which is very important; make no mistake), and consider a simple system that has mechanochemical coupling. We will look at a minimal system for mechanochemical coupling.

Consider a single chemical species, which we will call a "regulator." The regulator is embedded in the cortex and regulates active stress. It has concentration $c(x,t)$, and obeys a diffusion-advection equation,

$$\partial_t c = D\partial_x^2 c - \partial_x(vc), \tag{6.1}$$

where v is the fluid flow velocity in the x-direction. As in the Gross, et al. paper, we are only considering one dimension. This equation is akin to equation (3) in the Gross paper with the biochemical interactions stripped away.

The equation governing viscous cortical flow is

$$\partial_x\, \sigma_a = -\eta\, \partial_x^2 v + \gamma v, \tag{6.2}$$

which you derived in your previous homework and is also given by equation (6) of the Gross, et al. paper. We can already see that the two dynamical equations are coupled because the fluid flow velocity appears in both equations. The situation is more interesting because the active stress depends on the concentration of the regulator. We therefore define

$$\sigma_a = \sigma_a^0 f(c), \tag{6.3}$$

where $f(c)$ is a dimensionless function of c and σ_a^0 is the scale of the active stress. In the Gross, et al. paper, equation (6) gives that σ_a^0 can vary with time as the active stress component of the actomyosin cue, and

$$f(c) = c/(c + c_*). \tag{6.4}$$

For this problem, we will not need to assume any specific functional form for $f(c)$, so we will not.

a) We could use periodic boundary conditions, as in the Gross, et al. paper. It is a bit simpler to instead assume no-flux boundary conditions, i.e., $-D\partial_x c + vc = 0$ at $x = 0$ and $x = L$, and the same general results home. Using no-flux boundary conditions, discuss why or show that there exists a single homogeneous steady state, which we will arbitrarily call $c = c_0$.

b) Show that the steady state velocity is related to the steady state concentration by $v = D\partial_x \ln c$. This implies that a non-homogeneous profile has fluid flow, and also that there is no flow for the homogeneous steady state.

c) The **Péclet number**, Pe, is the ratio of the diffusive to advective time scales.

   i) Explain when $\mathrm{Pe} = \sigma_a^0 / D\gamma$ is a good definition of the Péclet number for this case.

   ii) If $\mathrm{Pe} \gg 1$, does advective or diffusive transport dominate the dynamics, and why?

d) We will now perform a linear stability analysis about the homogeneous quiescent steady state. The goal is to see if the system can spontaneously form a non-homogeneous pattern in $c(x)$ accompanied by flow.

   i) Consider a small perturbation to the homogenous steady state, $c = c_0 + \delta c$, with $\delta c = \delta c_0\, \mathrm{e}^{st+ikx}$, where k is the wave number of the spatial perturbation and $s = s(k)$. Show that to linear order in δc that

$$v(x) = \frac{ik\sigma_a^0 f_0}{\gamma + \eta k^2}\, \delta c, \tag{6.5}$$

   where $f_0 = \partial_c f(c_0)$, the value of the first derivative of $f(c)$ with respect to c, evaluated at c_0.

   ii) Show that the homogeneous steady state is unstable (and therefore patterns may spontaneously form) if

$$\frac{\mathrm{Pe}\, c_0 f_0}{1 + (\pi\ell/L)^2} > 1, \tag{6.6}$$

   where ℓ is the familiar hydrodynamic length scale from the Mayer and Gross papers, $\ell = \sqrt{\eta/\gamma}$. *Hint*: Because of the geometry of the system, allowed wave numbers are $k = n\pi/L$, where n is an integer.

   iii) What does equation (6.6) say about the requirements of the functional form of $f(c)$ in order to get patterns? Does the expression given in equation (6) for the contractility meet these requirements?

e) Presumably, the concentration of the active stress regulator can be controlled via regulation of gene expression. Other physical parameters like σ_a^0, η, and γ are not as easily controlled. In light of this, discuss the repercussions of equation (6.6) for how flowing patterns may be turned on and off.

f) Sketch a plot of a peak in concentration of the regulator along with the fluid flow velocity that we might see at steady state. It need not be quantitatively accurate; just a sketch will do. Based on this sketch, give a qualitative physical discussion on how patterns can be maintained through this active advection-diffusion mechanism.